

D-PARTICLE FIELD CATEGORY

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ABSTRACT. The homotopy Lie (L_∞) categorification of D -particle field theory in the method of gauged S -duality spontaneous breakdown is investigated. We show the non-trivial definition of the D -particle fields category and give explicit formulae for its higher composition structures. As a result, we can recognize the physical significance of D -particle fields for determinism and undeterminism.

1. INTRODUCTION

In this paper, we investigate the geometry of the D -particle ($D=0$ brane) field theory according to type IIA/ M theory.[1, 2] We adopt the 't Hooft's planar diagram theory of large N $SU(N)$ Yang-Mills theory for bounded open string fields to include the gravity/gauge correspondence and to reduce the degrees of freedom of Feynman diagrams [3]

There are several approaches to the constructive definition of superstrings and M theory.[4, 5] It has been recognized that they can be considered as second quantized theories of D -branes. Therefore, our present investigation can provide the basis of the geometrical foundation of a constructive definition of string theory.

Different constructive definitions of D -particle fields have been proposed by Yoneya and the author.[1, 2] In particular, the author has proposed D -brane field theory as a single $SL(2, \mathbb{R})$ BRST cohomology $\ker \mathbf{Q}$. The 3-vector partition function $\psi[g_s, \vec{t}, \vec{\bar{t}}]$ with deformation parameters t_n for n D -branes satisfies the equation

$$(1) \quad \left(i \sum_{I=1}^3 \Theta^I \left(Q \mathcal{A}^I + \frac{1}{2} \mathcal{R}^I \right) - \alpha' \Lambda \right) \psi[g_s, \vec{t}, \vec{\bar{t}}] = 0$$

where

$$(2) \quad \Theta^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Theta^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \Theta^3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The $SL(2, \mathbb{R})$ gauge potential \mathcal{A} and ghost \mathcal{C} for the coefficients of $SL(2, \mathbb{R})$ group factors ε^I are defined by

$$(3) \quad \mathcal{A} = \sum_{I=1}^3 \varepsilon^I \exp \left(\sum_{n=1}^{\infty} (s_n q_n^I + \bar{s}_n \bar{q}_n^I) \right), \quad \mathcal{C} = \sum_{I=1}^3 \varepsilon^I \exp \left(\sum_{n=1}^{\infty} (s_n \theta_n^I + \bar{s}_n \bar{\theta}_n^I) \right)$$

with nilpotent fermionic operators q_n^I (i.e, BRST charges for Chan-Paton unitary Yang-Mills theories) and Grassmann numbers θ_n^I . (We need to be careful of the index $I = 1, 2, 3$ of $SL(2, \mathbb{R})$ in q^I and θ^I , because these charges have meaning only

when they are summed over this index.) We define the quadratic variable as the external product between ghosts

$$(4) \quad \mathcal{R} = \mathcal{C} \times \mathcal{C}$$

and Λ is the eigenvalue of the Laplacian in the time variables t_n

$$(5) \quad \Delta \left[\vec{s}, \vec{s} \right] \mathcal{A} \Big|_{\vec{s}=\vec{s}=0} = -\Lambda \mathcal{A} \Big|_{\vec{s}=\vec{s}=0}, \quad \Delta \left[\vec{s}, \vec{s} \right] = \sum_{n=1}^{\infty} \left[\frac{\partial^2}{\partial s_n^2} + \frac{\partial^2}{\partial \bar{s}_n^2} \right].$$

If the gauged S -duality is not spontaneously broken, i.e., the classical solution is invariant under S -duality transformations, then Λ is exactly zero. So, the term $-\mathbf{1}\alpha'\Lambda\psi$ is regarded as the inducer of gauged S -duality spontaneous breakdown. Q is the sum of BRST charges of the $U(n)$ Chan-Paton factors q_n and \bar{q}_n

$$(6) \quad Q^I = \sum_{n=1}^{\infty} (q_n^I + \bar{q}_n^I).$$

This definition is equivalent to an $SL(2, \mathbb{R})$ Yang-Mills action with an infinite number of time variables, although we do not derive this equivalence here.

2. D-PARTICLE FIELD CATEGORY

2.1. Category of classical vacua Spaces. An L_{∞} category is an A_{∞} category whose composition structures satisfy L_{∞} relations.

We define an L_{∞} category such that its object set is the set of geodesics χ_1, χ_2, \dots on the Poincaré upper half plane \mathfrak{H} with an infinite number of time parameters

$$(7) \quad \mathbf{Spaces} = \bigoplus_{i=1}^{\infty} \chi_i [[T, \vec{s}]]$$

The time variables are the coefficients (growth rates) of the n -th differential of the motion of geodesics as parameterized by a single time variable T

$$(8) \quad \chi(T, \vec{s}) = \sum_{\ell=1}^{\infty} s_{\ell} \frac{\partial^{\ell} \chi(0)}{\partial T^{\ell}} T^{\ell}$$

The function $\chi(T)$ indicates either the center $O(T)$ of the circle or its radius $r(T)$.

The space of morphisms is defined as the following singular chain complex on **Spaces**

$$(9) \quad \mathcal{H}_*^{(0)}(\chi_1, \chi_2) = S_*(\chi_1, \chi_2)$$

for $* \leq 2$. In the following paragraphs, we suppress the time variables in each object.

To define the composition structures, we denote the angle of the geodesic between $i\infty$ and two complex numbers χ_1 and χ_2 as

$$(10) \quad \vartheta(\chi_1, \chi_2) = \arg \left(\frac{\chi_1 - \chi_2}{\chi_1 - \bar{\chi}_2} \right) = \frac{1}{2i} \ln \left(\frac{(\bar{\chi}_1 - \chi_2)(\chi_1 - \chi_2)}{(\chi_1 - \bar{\chi}_2)(\bar{\chi}_1 - \bar{\chi}_2)} \right)$$

The composition structure on **Spaces** is defined as the integral, precisely the summation of the integrals over differentiable intervals,

$$(11) \quad F(\chi) = \int_{\chi} d_{\chi} \vartheta,$$

where

$$(12) \quad d_\chi \vartheta = \lim_{\delta T \rightarrow +0} \vartheta(\chi(T), \chi(T + \delta T)) .$$

Higher-order composition structures are defined by

$$(13) \quad \mathfrak{l}_k^0(\Delta\chi_1, \dots, \Delta\chi_k) = \sum_{\chi_{k+1}} F\left(\sum_{m=1}^{k+1} \pm \Delta\chi_m\right) \mathbf{Space}(\chi_{k+1})$$

where the sign is positive for $1 \sim k$ and negative for $k+1$.

The nilpotency for **Spaces** is shown in the following way. We divide \mathfrak{H} using n geodesics $\chi_1, \chi_2, \dots, \chi_n$ and assign a common direction to each χ_i . Next, \mathfrak{H} is divided into cells, and

$$(14) \quad \sum_{1 \leq k, n \leq m} \sum_{\sigma \in \mathfrak{S}_m} \sum_{\sigma(\partial \mathfrak{C})} \pm \frac{1}{n!(m-n)!} \times \mathfrak{l}_{m-n}^0(\chi_1^+, \dots, \mathfrak{l}_n^0(\chi_k^+, \dots, \chi_{k+n-1}^+), \dots, \chi_m^+; \chi_{m+1}^-) = \sum_{\chi \in \partial \mathfrak{C}} \int_\chi d\vartheta^{(0)} = 0$$

where σ is a faithful action from symmetric group \mathfrak{S}_m with m elements and $\vartheta^{(0)}$ has no time variable and \mathfrak{C} is the set of cells on the divided area. The boundary on \mathfrak{H} is $i\infty^- \cup \mathbb{R}^+$. When we define the integral under the modulo $2\pi\infty$, due to the Stokes theorem the result is zero.

2.2. Category of quantum vacua $\ker \mathbf{Q}$. The category of D -particle fields is defined as follows. The L_∞ structure of it is mathematically a conjecture.

The objects are elements of the kernel of the $SL(2, \mathbb{R})$ BRST charge \mathbf{Q}

$$(15) \quad \ker \mathbf{Q}$$

such that the $SL(2, \mathbb{R})$ symmetry of each object breaks to $SL(2, \mathbb{Z})$. Each element, as a cusp form for a Fuchsian subgroup of $SL(2, \mathbb{Z})$, has a Fourier expansion due to the periodicity $\ln \psi = \ln \psi + 2\pi i$:

$$(16) \quad \psi = \sum_{n=0}^{\infty} \tau_n \psi^n .$$

The saddle points of ψ are given by

$$(17) \quad \left. \frac{\delta \psi}{\delta |\chi|} \right|_{\chi=\chi^{(0)}} = 0, \quad \left. \frac{\delta^2 \psi}{\delta |\chi|^2} \right|_{\chi=\chi^{(0)}} > 0$$

The saddle points are denoted as the absolute value of the variable $|\chi|$ on \mathfrak{h} and form an infinite sequence

$$(18) \quad |\chi_1^{(0)}| < |\chi_2^{(0)}| < \dots$$

including infinity, $\chi_\infty^{(0)} = i\infty$. These saddle points

$$(19) \quad \chi_\ell^{(0)} \left[\vec{s}, \vec{s}, \Lambda \right] \simeq \frac{\sqrt{\alpha'}}{u_\ell} \omega_\ell, \quad (\omega_\ell)^\ell = 1$$

for functions with length dimension $u_\ell(\vec{s}, \vec{s}, \Lambda)$ of parameters in the BRST equation correspond to the classical distribution of ground states. Here, the equality $a \simeq b$ indicates that a and b are in the same conjugacy class of $SL(2, \mathbb{Z})$.

We define the ideal \mathbb{I} of cusps as $\ell\mathbb{Z}$.

The morphisms are defined by correlations of Goldstone modes $\tilde{\psi}$ that satisfy the BRST condition $\mathbf{Q}\tilde{\psi} = 0$ locally.[1] The BRST transformation is a global one and, by this transformation, each Goldstone mode $\tilde{\psi}$ draws a geodesic on \mathfrak{h} that passes through $\chi^{(0)}$. In particular, the line which passes the first cusp $i\infty$ represents gravity. We denote these geodesics as C .

The diagram of the cluster of n D -particles on the ‘master’ upper half plane $\mathfrak{H}[g_s]$ is given by geodesics connecting one point of a geodesic through $\chi_m^{(0)}$ and one point of another geodesic through $\chi_{m\pm n}^{(0)}$. The points $\chi_\ell^{(0)}[s_\ell, \bar{s}_\ell, \Lambda]$, representing ℓ D -particles on their own geodesics, promote the time parameter s_ℓ by an $SL(2, \mathbb{R})$ global BRST transformation.

At each vertex on a diagram of D -particles, the total number of ghosts n and the sum of coefficients p_n for the BRST isometry transformation $\mathbf{U}_n = p_n \partial / \partial t_n$ are conserved.

The bounded open string fields are drawn as an infinite number of geodesics as the Goldstone modes, the solution of variations $\delta\psi = 0$ or, equivalently, $q_n C[t_\ell, \bar{t}_\ell]_{\ell=1}^n = 0$ for $n = 1, 2, \dots$. Although they are drawn on each upper half plane $\mathfrak{h}_n[\tau[t_n, \bar{t}_n]]$ with a time promotion parameter t_n , they are matched by the coupling constant g_s as the same event geodesic on the master upper half plane $\mathfrak{H}[g_s]$. These motions of geodesics include the non-trivial one since the $SL(2, \mathbb{R})$ symmetry is a gauge symmetry.

There are elements of the BRST cohomology of Chan-Paton open string fields bounded on D -particle fields ψ_1 and ψ_2 quotiented by the set of critical points Cr (vacuum configurations) of the gauge potential \mathcal{A}

$$(20) \quad \mathcal{H}(\psi_i^{\vec{n}_1, \vec{m}_1}, \psi_j^{\vec{n}_2, \vec{m}_2}) = \begin{cases} \bigoplus_{\vec{n}} \left(H^1(q_{n_1}) / Cr \right) & \vec{n}_1 = \vec{n}_2 = \vec{m}_1 = \vec{m}_2 \\ 0 & otherwise \end{cases}$$

We note that $\widetilde{\psi_{i,i}} = \psi_i$. Here indices \vec{n} and \vec{m} denote the indices of Grassmann numbers $\prod_{\vec{n}} \theta_n$ and $\prod_{\vec{m}} \bar{\theta}_m$. These cohomologies satisfy the ring structures of quantum cohomologies.

It is natural from a physical point of view to assume that the higher composition structure $C_{i,j} \in \mathcal{H}(\psi_i, \psi_j)$ between k geodesics is defined as the product of the parts of the physical state describing the interaction between two geodesics:

$$(21) \quad \mathfrak{l}_k(C_{1,2}, C_{2,3}, \dots, C_{k,k+1}) = \sum_{C_{1,k+1}} \mathfrak{l}_{1,2,\dots,k;k+1} C_{1,k+1}$$

where $\mathfrak{l}_{1,2,\dots,k;k+1}$ is explicitly given by the multi- or infinite-dimensional integral

$$(22) \quad \mathfrak{l}_{1,2,\dots,k;k+1} = \sum_{i_2, i_3, \dots, i_k} \prod_{\ell=1}^k \psi_{i_\ell, i_{\ell+1}} \left[g_s, \vec{t}, \vec{\bar{t}} \right].$$

The physical state is defined in the language of geodesics and critical points and explicitly given by the propagators of open string fields and one of D -particles¹

$$(23) \quad \psi[g_s, \vec{t}, \vec{t}] = \sum_{(i,j)} \sum_{\ell=1}^{\infty} \sum_{\gamma \in [\partial \mathfrak{C}_i \cap \partial \mathfrak{C}_j]_{\ell}} g_s^{([\mathbb{I}_i \cap \mathbb{I}_j]_{\ell})} \mathscr{W}_{\gamma} \int_{\mathcal{M}^{\ell}} \bigwedge_1^{\ell} d\vartheta_{\ell}(\chi_i, \chi_j) \times \left[\left[\prod_1^{\ell} \theta \right] \left[\prod_1^{\ell} \bar{\theta} \right] \right]_{\mathbb{I}_i \cap \mathbb{I}_j}$$

for $d\vartheta = d_{\chi_i}\vartheta + d_{\chi_j}\vartheta$ and the classical landscape [6, 7] $\mathbf{Spaces} = \bigoplus_{\ell=1}^{\infty} \mathcal{M}^{\ell}$ of vacua $\chi[[\vec{s}, \vec{s}, \Lambda]]$ deformed by parameters in BRST equation.

The propagator of open string fields \mathscr{W}_{γ} for Poincaré metric g_{IJ} , $I, J = 1, 2$ on the upper half plane is defined using the result of Poisson type σ model as [8, 9]

$$(24) \quad \mathscr{W}_{\gamma} = \sum_{i,j} \left[\left[\prod_{\gamma} \partial_{\partial_+ \gamma} \right] C_i \times \left[\prod_{\gamma} \partial_{\partial_+ \gamma} \right] C_j \right] \left[\prod_{(\gamma_1, \gamma_2)} \left[\prod_{I_{1,2} \in \gamma_1 \cap \gamma_2} \partial_{I_{1,2}} \right] g^{\partial_- \gamma_1 \partial_- \gamma_2} \right]$$

for $\partial_-(\gamma) = \gamma(0)$ and $\partial_+(\gamma) = \gamma(+\infty)$. We alert that γ is not a geodesic but just an interval. If $\gamma \cap C = 0$, then we rule $\partial_{\partial_{\pm} \gamma} C = C$.

Its (i, j) components are

$$(25) \quad \psi = \sum_{(i,j)} \psi_{i,j} .$$

Next, we state the following conjecture:

Conjecture 1. *The above composition structure \mathfrak{l} satisfies nilpotency. Therefore, the above ‘category’ $\ker \mathbf{Q}$ is an L_{∞} category.*

The ‘ L_{∞} category’ $\ker \mathbf{Q}$ defines the Hochschild cohomology of $\mathfrak{l}^2 = 0$

$$(26) \quad HH(\mathbf{Spaces}) = H^1(\mathcal{L}), \quad \mathcal{L} = \mathfrak{l} \cdot \mathcal{F}$$

for an L_{∞} functor \mathcal{F} from \mathbf{Spaces} to $\ker \mathbf{Q}$.

Next, we state a second conjecture:

Conjecture 2. *There exists an exact sequence between \mathbf{Spaces} and $\ker \mathbf{Q}$.*

$$(27) \quad 1 \longrightarrow \mathbf{Spaces} \longrightarrow \ker \mathbf{Q} \longrightarrow HH(\mathbf{Spaces}) \longrightarrow 1 .$$

If conjecture 1 holds true, this statement is obvious due to the fact that the L_{∞} category $\ker \mathbf{Q}$ decomposes into $\prod_{\ell=1}^{\infty} \mathcal{M}^{\ell} \otimes \mathfrak{h}_{\ell}$. In this case, $\ker \mathbf{Q} = \mathbf{Spaces} \oplus HH(\mathbf{Spaces})$ holds.

3. PHYSICAL INTERPRETATION

The physical interpretation of the coefficients \mathfrak{l} is that they are probability coefficients in the superposition of D -particle fields after two D -particle fields interact

$$(28) \quad \mathfrak{l}_2(C_1, C_2) = \sum_{C_3} \mathfrak{l}_{1,2,3} C_3 .$$

¹In the following paragraphs, the index ℓ of $d\vartheta_{\ell}$ indicates the number of D -particles as described by $d\vartheta$.

The nilpotency $l^2 = 0$ is physically necessary since it is the consistency condition of ‘unitarity’ for D -particle fields and our interpretation can be realized only when we take into consideration the field theory of D -branes.

If our theory is consistent, D -particle field theory is deterministic and there is no place for indeterministic axioms, such as the collapse of a superposition. There is a double structure of probabilities: one structure of D -brane fields and another of (planar) open string fields. So, the indeterminism of the latter is reduced to the determinism of the former. The reduction of indeterminism to determinism results solely from the existence of one bounded geodesic on the tiling decomposition of the infinite dimensional space $\mathfrak{C}[[\vec{t}, \vec{t}]]$ together with the unitarity of D -particle fields.

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